

# Introduction to Probability & Statistics

## Assignment 3, 2025/26

### Instructions

Submit your answers to the **four questions marked  Hand-in**. You should upload your solutions to the VLE as a *single pdf file*. Marks will be awarded for clear, logical explanations, as well as for correctness of solutions.

Solutions to questions marked  have been released at the same time as this assignment, in case you want to check your answers or need a hint.

You should also look at the other questions in preparation for your Week 7 seminar.

### Starters

*These questions should help you to gain confidence with the basics.*

**S1.**  Each morning a student rolls a die and starts studying if she gets a 6. Otherwise, she stays in bed. However, during the four months of exams, she tosses a coin instead of rolling a die and studies if she gets a Head. On a (randomly chosen) morning, the student is studying. What is the probability that it is exam period? (Assume for simplicity that each month has the same number of days!)

**S2.**  Consider a discrete random variable  $V$  taking values in  $\{1, 2, 3, 4\}$  with mass function  $p_V$ :

$$p_V(x) = \begin{cases} xc^x & \text{if } x = 1, 2 \\ c(5 - x) & \text{if } x = 3, 4. \end{cases}$$

What value of  $c$  makes  $p_V$  a mass function?

**S3.**  **Hand-in**

Consider a discrete random variable  $Y$  taking values in  $\{1, 2, 3, 4, 5\}$  with mass function  $p_Y$  given by

$$p_Y(y) = \begin{cases} cy^2 & \text{if } y = 1, 2 \\ (6 - y)/8 & \text{if } y = 3, 4, 5. \end{cases}$$

What value of  $c$  makes  $p_Y$  a mass function?

### Answer

To be a mass function, we need  $p_Y(y) \geq 0$  for all  $y$  and  $\sum_y p_Y(y) = 1$ . **[2 marks]** The first is satisfied if  $c \geq 0$ . For the second constraint we need  $c(1^2 + 2^2) + (6 - 3)/8 + (6 - 4)/8 + (6 - 5)/8 = 1$ , and so  $c = 1/20$ . **[3 marks]**

**S4.** What is the probability that exactly 3 heads are obtained in 5 tosses of a fair coin?

**S5.**  A fair die is thrown until the sum of the results of the throws exceeds 6. The random variable  $X$  is the number of throws needed for this. Let  $F_X$  be the distribution function of  $X$ . Determine  $F_X(1)$ ,  $F_X(2)$  and  $F_X(7)$ .

**S6.** ✂ The probability density function  $f_X$  of a continuous random variable  $X$  is given by

$$f_X(x) = \begin{cases} x(3-x) + c & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Compute  $c$ . Work out the distribution function of  $X$ .

**S7.** 📁 **Hand-in**

The probability density function  $f_X$  of a continuous random variable  $X$  is given by

$$f_X(x) = \begin{cases} cx^2 & \text{if } 0 \leq x \leq 1 \\ 1/2 & \text{if } 2 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

Compute  $c$ , and hence calculate  $\mathbb{P}(X > 1/2)$ .

Answer

The constant  $c$  is determined by the requirement that the integral of the density function over the entire real line is equal to one:

$$1 = \int_{-\infty}^{\infty} f_X(x) dx = \int_0^1 cx^2 dx + \int_2^3 \frac{1}{2} dx = \frac{1}{2} + \frac{c}{3}.$$

This implies that  $c = 3/2$ . Note that for this value of  $c$  the function  $f_X$  is non-negative everywhere, which is also required in order for it to be a density. **[3 marks]**

We can then calculate

$$\mathbb{P}(X > 1/2) = 1 - \mathbb{P}(X \leq 1/2) = 1 - \int_0^{1/2} \frac{3x^2}{2} dx = \frac{15}{16}.$$

**[2 marks]**

**S8.** Let  $Y \sim \text{Poisson}(2)$ . Write down the mass function for  $Y$ . What is  $\mathbb{P}(Y \geq 2)$ ?

**S9.** A continuous random variable  $X$  has a distribution function  $F$  which satisfies

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{4x^2+3x}{7} & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1. \end{cases}$$

Determine  $\mathbb{E}[X]$ .

## Mains

*These are important, and cover some of the most substantial parts of the course.*

**M1.** 📁 **Hand-in**

Consider the probability space corresponding to throwing a fair die. Give an example of a random variable  $X$  whose image is  $\{1, 2, 3\}$  and for which  $F_X(2.7) = 5/6$ .

Answer

There are lots of possibilities. The only important thing is that exactly one of the outcomes ( $\Omega = \{1, 2, 3, 4, 5, 6\}$ ) is mapped to 3 so that  $F_X(2.7) = \mathbb{P}(X \leq 2.7) = 1 - \mathbb{P}(X = 3) = 1 - 1/6 = 5/6$ . One possibility is  $X(1) = 1, X(2) = 1, X(3) = 1, X(4) = 2, X(5) = 2, X(6) = 3$ . **[3 marks for a correct example; 2 for some sort of reasoning]**

**M2** 🔄 **Hand-in**

A box contains four coins: one has a Head on both sides; one has a Tail on both sides; the other two coins are normal and fair. A coin is chosen at random from the box and tossed three times. What is the probability that the two-Tailed coin was chosen, given that all three tosses are Tails?

Answer

Let  $A_1 = \{\text{two-Headed coin chosen}\}$ ,  $A_2 = \{\text{two-Tailed coin chosen}\}$ ,  $A_3 = \{\text{fair coin chosen}\}$ , and  $B = \{\text{obtain three Tails}\}$  [1 mark for setting up some notation].

Then

$$\mathbb{P}(B) = \sum_{i=1}^3 \mathbb{P}(B | A_i) \mathbb{P}(A_i) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + \left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} = \frac{5}{16}.$$

[2 marks] Finally, we can use Bayes' theorem:

$$\mathbb{P}(A_2 | B) = \frac{\mathbb{P}(B | A_2) \mathbb{P}(A_2)}{\mathbb{P}(B)} = \frac{1 \cdot 1/4}{5/16} = \frac{4}{5}.$$

[2 marks]

**M3.** ✂ Let  $\lambda > 0$  be some positive number. Find  $c$  such that the following function is a density function:

$$f(x) = ce^{-\lambda|x-1|}, \quad x \in \mathbb{R}.$$

**M4.** ✂ Recall the Chevalier de Méré's problem: two dice are thrown 24 times – you win £1 if at least one double 6 is thrown, otherwise you lose £1. Show that you expect to lose money on average if you play this game.

**M5.** Suppose that  $A$  and  $B$  are independent events. Show that  $A^c$  and  $B^c$  are independent.

**M6.** Consider a roulette wheel, with numbers 00, 0, 1, 2, 3, ..., 36 (38 numbers in total). A ball is thrown onto the wheel as it is spinning, and comes to rest by one of the numbers. You always bet that the ball will stop on one of the numbers 1, 2, ..., 12. Let  $N$  be the random variable giving the number of bets that you lose *before* your first win. Calculate  $p_N(0)$ ,  $p_N(5)$  and  $F_N(5)$ .

**M7.** Consider a discrete random variable  $Y$  taking values in the set  $\{0, 1, \dots, 8\}$  and with probability mass function  $p_Y$  given by

$$p_Y(k) = \begin{cases} ca & \text{if } k = 0, 1, 2, 3, 4, 5 \\ ca^2 & \text{if } k = 6, 7 \\ c(1-a)^2 & \text{if } k = 8, \end{cases}$$

where  $a$  is some fixed number between 0 and 1 and  $c$  is a constant to be determined. What value of  $c$  makes  $p_Y$  a mass function? (The answer is a function of  $a$ .)

**M8.** Let  $X \sim \text{Bern}(p)$ . Let  $Y = 1 - X$  and  $V = X^2$ . Show that

- $Y \sim \text{Bern}(1 - p)$ ;
- $V \sim \text{Bern}(p)$ .

**M9.**  $N \geq 3$  people go for coffee. Each person flips a fair coin: if all but one of the coins shows the same face (Head or Tail), then the odd person out pays for all the drinks; if not then the coins are tossed again until this event occurs. How many times on average must each person toss their coin before somebody is selected in this way?

**M10.** Prove the memoryless property of the exponential distribution which states that if  $X \sim \text{Exp}(\lambda)$  then for any  $t, s \geq 0$ ,

$$\mathbb{P}(X > t + s | X > s) = \mathbb{P}(X > t).$$

## Desserts

*Still hungry for more? Try these if you want to push yourself further. (These are mostly harder than I'd expect you to answer in an exam, or involve non-examinable material.)*

**D1.**  $N \geq 3$  people go for coffee. Each person flips a coin, of which  $N - 1$  are fair, and one has probability  $q$  of coming up Heads (for some  $q \in [0, 1]$ ): if all but one of the coins shows the same face (Head or Tail), then the odd person out pays for all the drinks; if not then the coins are tossed again until this event occurs. How does the expected number of coin tosses each person makes vary with  $q$ ?

**D2.** Alice and Bob decide to duel, using just one six-shot revolver, and one bullet, between them. They decide to duel in the following way: with the bullet inserted into the revolver, Alice will spin the cylinder and shoot at Bob (killing him if the gun fires); if Alice misses, Bob will spin the cylinder and shoot at Alice. Assuming there is a  $1/6$  probability that the revolver will fire each time (since the revolver has 6 chambers), what is

- the distribution of the number of turns  $T$  until the gun fires?
- the probability that Alice wins the duel?

### ! Challenge question

At time 0, a bag contains 1 Red and 1 Black ball. Just before each time  $n = 1, 2, 3, \dots$ , a ball is chosen at random from the bag and then replaced, *along with a new ball of the same colour*. For  $n = 0, 1, 2, \dots$  and  $1 \leq r \leq n + 1$ , let

$$p_{n,r} = \mathbb{P}(\text{at time } n, \text{ the bag contains exactly } r \text{ Red balls}) .$$

- Write down the values of  $p_{0,1}$ ,  $p_{1,1}$  and  $p_{1,2}$ ;
- By conditioning on the possible numbers of red balls at time 1, calculate  $p_{2,r}$  for  $r = 1, 2, 3$ ;
- Can you find (and prove!) a general formula for  $p_{n,r}$ ? [HINT: guess at the right answer using your findings from parts (a) and (b). (Calculate  $p_{3,r}$  explicitly if you can't see a pattern.) Then use induction.]