Introduction to Probability & Statistics

Assignment 2, 2025/26

Instructions

Submit your answers to the **four questions marked Hand-in**. You should upload your solutions to the VLE as a *single pdf file*. Marks will be awarded for clear, logical explanations, as well as for correctness of solutions.
Solutions to questions marked have been released at the same time as this assignment, in case you want to check your answers or need a hint.
You should also look at the other questions in preparation for your Week 5 seminar.

Tip

Make sure to justify your reasoning by making reference to any of the properties (P1)–(P7) covered in lectures, where you are using them in your answers.

Starters

These questions should help you to gain confidence with the basics.

S1. Let E, F and G be three events. Here are five events built from these:

a) $E \cap F^c \cap G^c$
b) $(E \cup F) \cap G^c$
c) $E \cup F \cup G$
d) $(E \cap F \cap G)^c$
e) $(E \cap F)^c \cap (E \cap G)^c \cap (F \cap G)^c$.

And here are verbal descriptions of the same five events, but in a different order:

1) E or F occurs, but not G
2) At most two of the events occur
3) At most one of the events occurs
4) At least one of the events occurs
5) Only E occurs.

Match the expressions to the verbal descriptions.

S2. Let C and D be events. Express the probability $\mathbb{P}(C^c \cap D)$ in terms of $\mathbb{P}(D)$ and $\mathbb{P}(C \cap D)$.

S3. Let E and F be two events for which we know that the probability that at least one of them occurs is 0.4. What is the probability that neither E nor F occurs?

S4. Calculate $\mathbb{P}(B)$ if it is given that $\mathbb{P}(A \cup B) = 3/7$ and $\mathbb{P}(A^c \mid B^c) = 9/14$.

S5. Hand-in
When $\mathbb{P}(A) = 1/4, \mathbb{P}(B) = 2/3$, and $\mathbb{P}(A \cup B) = 3/4$, calculate $\mathbb{P}(A \cap B)$ and $\mathbb{P}(A^c \cup B^c)$.

Answer

Solving the relation $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ (property (P6)) for $\mathbb{P}(A \cap B)$ gives

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 1/4 + 2/3 - 3/4 = 1/6$$

[2 marks]. Using De Morgan's law, and then (P4), we get

$$\mathbb{P}(A^c \cup B^c) = \mathbb{P}((A \cap B)^c) = 1 - \mathbb{P}(A \cap B) = 1 - 1/6 = 5/6.$$

[3 marks]

Mains

These are important, and cover some of the most substantial parts of the course.

M1. Hand-in
Let E and F be two events. Prove that if $E \subseteq F$ then

$$\mathbb{P}(F \cap E^c) = \mathbb{P}(F) - \mathbb{P}(E).$$

Answer

Since $E \subseteq F$, we can write F as the *disjoint* union $F = (F \cap E^c) \cup E$. **[2 marks]** This gives

$$\mathbb{P}(F) = \mathbb{P}((F \cap E^c) \cup E) \overset{(P3)}{=} \mathbb{P}(F \cap E^c) + \mathbb{P}(E).$$

Rearranging gives the required result. **[3 marks]**

M2. Hand-in
Calculate $\mathbb{P}(A \cup B)$ if it is given that $\mathbb{P}(B) = 1/5$ and $\mathbb{P}(A \,|\, B^c) = 1/3$.

Answer

We use our usual trick of rewriting a union in terms of the union of disjoint events. In this case the best is to write $A \cup B = B \cup (A \cap B^c)$ and thus, by (P3), $\mathbb{P}(B \cup A) = \mathbb{P}(B) + \mathbb{P}(A \cap B^c)$. We can then use that by the multiplication rule

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(A|B^c)\,\mathbb{P}(B^c)$$

[2 marks]. Furthermore $\mathbb{P}(B^c) = 1 - \mathbb{P}(B)$, by (P4). Putting everything together gives

$$\begin{aligned}\mathbb{P}(B \cup A) &= \mathbb{P}(B) + \mathbb{P}(A \,|\, B^c)\,(1 - \mathbb{P}(B)) \\ &= \frac{1}{5} + \frac{1}{3}(1 - 1/5) = \frac{7}{15}.\end{aligned}$$

[3 marks]

M3. Show that if an event occurs with probability 0 then it is independent of all other events.

M4. Prove that, for any events A, B, C for which the following probabilities are all defined,

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B \,|\, A)\,\mathbb{P}(C \,|\, A \cap B).$$

M5. Hand-in

Let A, B, and C be events. Prove that if

$$\mathbb{P}(A \mid C) > \mathbb{P}(B \mid C) \text{ and } \mathbb{P}(A \mid C^c) > \mathbb{P}(B \mid C^c)$$

then $\mathbb{P}(A) > \mathbb{P}(B)$.

Answer

We can use the partition theorem **[2 marks]**:

$$\mathbb{P}(A) = \mathbb{P}(A \mid C)\mathbb{P}(C) + \mathbb{P}(A \mid C^c)\mathbb{P}(C^c) > \mathbb{P}(B \mid C)\mathbb{P}(C) + \mathbb{P}(B \mid C^c)\mathbb{P}(C^c) = \mathbb{P}(B).$$

[3 marks]

M6. Show that if an event A is independent of itself, then either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

M7. In a card game involving four players, each player is dealt four cards (face down), and the player with the highest Diamond wins (Ace counts high). Upon receiving your four cards you see that your highest Diamond is the Queen of Diamonds. What is the probability that another player has the King or Ace of Diamonds, and hence that you lose the game?

M8. Consider a sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ consisting of four equiprobable elements. Let $A_1 = \{\omega_1, \omega_2\}$, $A_2 = \{\omega_1, \omega_3\}$, $A_3 = \{\omega_1, \omega_4\}$. Show that A_1 and A_2 are independent, A_1 and A_3 are independent, and A_2 and A_3 are independent. Are A_1, A_2, A_3 independent?

Desserts

Still hungry for more? Try these if you want to push yourself further. (These are mostly harder than I'd expect you to answer in an exam, or involve non-examinable material.)

D1. Given a sample space Ω, let $\mathbb{Q}$ be some function satisfying

$$\mathbb{Q}(A_1 \cup A_2) = \mathbb{Q}(A_1) + \mathbb{Q}(A_2)$$

for any two disjoint subsets A_1 and A_2 of Ω.

Show by induction that if $A_1, ..., A_n$ are $n \geq 2$ disjoint subsets of Ω then

$$\mathbb{Q}(A_1 \cup ... \cup A_n) = \mathbb{Q}(A_1) + ... + \mathbb{Q}(A_n) .$$

! Challenge question

Prove the **law of inclusion-exclusion** for three possibly overlapping sets A_1, A_2, A_3,

$$\begin{aligned}\mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) \\ - \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_3) - \mathbb{P}(A_2 \cap A_3) \\ + \mathbb{P}(A_1 \cap A_2 \cap A_3) .\end{aligned}$$

Can you prove the more general version? For sets $A_1, \ldots, A_n$,

$$\begin{aligned}\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i_1 < i_2} \mathbb{P}(A_{i_1} \cap A_{i_2}) + \ldots \\ + (-1)^{k+1} \sum_{i_1 < i_2 < \ldots < i_k} \mathbb{P}(A_{i_1} \cap A_{i_2} \ldots \cap A_{i_k}) \\ + \ldots + (-1)^{n+1} \mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) .\end{aligned}$$